Expectation of the Ratio of the Sum of Squares to the Square of the Sum: Exact and Asymptotic Results

Article *in* Theory of Probability and Its Applications · January 2002

DOI: 10.1137/S0040585X97978919

Expectation of the Ratio of a Sum of Squares to the Square of the Sum : Exact and Asymptotic results

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2630

October 1999

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Abstract

Let $X_i, i = 1 \dots n$ be a sequence of positive i. i. d. random variables. Define

$$R_n := E \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2}.$$

Let $\phi(s) = Ee^{-sX}$. We give an explicit representation of R_n in terms of ϕ and with the help of the Karamata theory of functions of regular variation we study the asymptotic behaviour of R_n for large n.

Résumé

Soient $X_i, i = 1 \dots n$ une suite de variables aléatoires positives équidistribuées,

$$R_n := E \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2},$$

et $\phi(s) = Ee^{-sX}$. Nous donnons une expression explicite de R_n à l'aide de ϕ et grâce à la théorie de Karamata des fonctions à variation régulière nous décrivons le comportement asymptotique de R_n pour les grandes valeurs de n.

1. Introduction. —

Let $X_1, X_2 \dots X_n \dots$ be a sequence of identically distributed independent non-negative random variables satisfying $P(X_1 = 0) < 1$. Let

$$R_n := E\frac{X_1^2 + X_2^2 + \dots + X_n^2}{(X_1 + X_2 + \dots + X_n)^2} \quad \text{with the convention} \frac{0}{0} = 0.$$

The asymptotic behaviour of R_n has been investigated in [CH] and [MO].

In section 2 an explicit formula is established representing R_n in terms of the Laplace transform φ of the X_i : $\varphi(s) := Ee^{-sX_1} = \int_0^\infty e^{-sx} dF(x), s \geq 0$, where F denotes the distribution function of X_1.

In section 3 the power of the representation formula is illustrated by giving elementary proofs of known results.

The representation formula allows us both to provide new and simpler proofs of known results and to extend them. For instance theorem (5.4) gives a new characterisation of laws which are in the domain of attraction of a stable law of parameter $\alpha \in (0,1)$. Our method relies on the theory of functions of regular variation for which [BGT] is our main reference. In the appendix we summarize the results we use from [BGT]; given these, this article is completely self contained.

In sections 4 and 5 several theorems are stated concerning the asymptotic behaviour of R_n. In section 4 these are of a general nature, section 5 gives a complete description of the behaviour of R_n when the X_i are the domain of attraction of a stable law. Theorems 4.1 and 4.2 are slight improvements of theorem 1 of [MO], theorems 5.1 and 5.2 are the theorem 2. of [MO], theorem 5.5 is theorem 1 of [CH] and theorem 4 of [MO], the first part of theorem 5.4 is the first part of theorem 4 of [CH],whose proof is omitted, and the second part of theorem 5.4 and theorem 5.6 are new. Most of these results were stated without proof in [FJ] though there is a small incongruity in their theorem 1.

2. Representation formulae. —

Let X and Y be non-negative random variables with Laplace transforms $\varphi_X(s), \varphi_Y(s)$ and joint Laplace transform $\varphi_{X,Y}(s,t)$. The formula

$$\frac{1}{y^\alpha} = \frac{1}{(\alpha)} \int_0^\infty e^{-sy} s^{\alpha-1} ds \ , \ \alpha > 0,$$

and Fubini's theorem yields

$$E\frac{1}{Y^\alpha} = \frac{1}{(\alpha)} \int_0^\infty s^{\alpha-1} \varphi_Y(s) ds \geq 0,$$

and for k_1 and k_2 non-negative integers :

$$E\frac{X^{k_1} Y^{k_2}}{(X+Y)^\alpha} = (-1)^{k_1+k_2} \frac{1}{(\alpha)} \int_0^\infty s^{\alpha-1} \frac{\partial^{k_1+k_2}}{\partial^{k_1} s \ \partial^{k_2} t} \varphi_{X,Y}(s,t)|_{s=t} ds.$$

In the special case when $X_1, \dots, X_n$ are independent random variables this gives

$$E\frac{X_1^k}{(X_1 + \dots + X_n)^\alpha} = \frac{(-1)^k}{(\alpha)} \int_0^\infty s^{\alpha-1} \varphi_{X_1}^{(k)}(s) \prod_2^n \varphi_{X_i}(s) ds, \ \alpha > 0,$$

and finally, when they have the same distribution, this yields by symmetry the representation formulae :

$$E\frac{X_1^k + \dots + X_n^k}{(X_1 + \dots + X_n)^\alpha} = n\frac{(-1)^k}{(\alpha)} \int_0^\infty s^{\alpha-1} \varphi_{X_1}^{(k)}(s) \varphi_{X_1}^{n-1}(s) ds \leq \infty$$

$$R_n = E\frac{X_1^2 + \dots + X_n^2}{(X_1 + \dots + X_n)^2} = n \int_0^\infty s \varphi_{X_1}^{''}(s) \varphi_{X_1}^{n-1}(s) ds. \tag{2.1}$$

This last formula is relevant for our problem.

3. Some elementary results. —

Let us return to the initial situation in which the X_i are i. i. d. non negative r. v. . There common Laplace transform is denoted by φ and the starting point of most of the proofs is the above representation formula (2.1). Proofs of the following lemmas are easy and are left to the reader. Let

$$\mu := EX_1 \text{ if } EX_1 < \infty, \quad d(s) := \frac{1-\varphi(s)}{s}.$$

LEMMA 3.1 2. — *If* $E(X_1^2) < \infty$, *then* $\lim_{n\uparrow\infty} nR_n = \frac{E(X_1^2)}{[E(X_1)]^2}$.

PROOF. It suffices, from lemma 3.5, to establish the result for the expression :

$$nA_n = n^2 \int_0^{\epsilon} s\varphi^{''}(s)\varphi^{n-1}(s)ds = \int_0^{n\epsilon} t\varphi^{''}(\frac{t}{n})\varphi^{n-1}(\frac{t}{n})dt.$$

With the help of Lemma 3.4, the integrand is seen to converge to $tE(X_1^2)e^{-\mu t}$. Using Lemma 3.3 and the monotonocity of φ and φ', the result follows by Lebesgue's dominated convergence theorem. ∎

THEOREM 2. — *If* $EX_1 < \infty$, *then* $\lim_{n\uparrow\infty} R_n = 0$.

PROOF. It suffices,from lemma 3.5, to establish the result for A_n. Lemma 3.1 implies that the integrand converges to 0. The result follows by Lebesgue's dominated convergence theorem since the assumption implies that $\varphi^{''}(s)$ is integrable and with the help of Lemma 3.3, the following upper bound is easily obtained, for $s \le s_0$

$$ns\varphi^{n-1}(s) \le \frac{ns\varphi^n(s)}{\varphi(\epsilon)} \le \frac{-ns\varphi'(s)e^{ns\varphi'(s)}}{-\varphi(\epsilon)\varphi'(\epsilon)} \le \frac{e^{-1}}{-\varphi(\epsilon)\varphi'(\epsilon)}.$$

∎

4. Asymptotic behaviour of R_n : general results. —

The next result is a slightly more precise version of Theorem 1 of [MO] :

THEOREM 2. —

$$\limsup_{n\to\infty} \frac{R_n}{\frac{1}{n}\int_0^n x[1-F(x)]dx} \le \begin{cases} 2 & \textit{if } 1 < EX_1 \le \infty \\ \dfrac{2}{\mu^2} & \textit{if } \mu := EX_1 \le 1 \ . \end{cases}$$

PROOF. It suffices from lemma 3.5 to establish the results for A_n. That φ is decreasing implies

$$A_n \le \frac{n}{\varphi(\epsilon)} \int_0^{\epsilon} s\varphi^{''}(s)\varphi^n(s)ds,$$

hence from lemma 3.3

$$A_n \le \frac{n}{\varphi(\epsilon)} \int_0^{\epsilon} s\varphi^{''}(s)e^{-nsd(\epsilon)}ds$$

$$\le \frac{n}{\varphi(\epsilon)} \int_0^{\infty} x^2 dF(x) \int_0^{\infty} se^{-s(x+nd(\epsilon))}ds = \frac{1}{\varphi(\epsilon)}(D_n + E_n),$$

where

$$D_n = n\int_0^n \left(\frac{x}{x+nd(\epsilon)}\right)^2 dF(x),\ E_n = n\int_n^\infty \left(\frac{x}{x+nd(\epsilon)}\right)^2 dF(x).$$

a) $E_n \le n\int_n^\infty dF(x) = n[1-F(n)]$,

$$\text{b) } D_n \le \frac{n}{[nd(\epsilon)]^2}\int_0^n x^2 dF(x) \le \frac{-n}{[nd(\epsilon)]^2}\int_0^n x^2 d[1-F(x)]$$
$$= \frac{2}{nd(\epsilon)^2}\int_0^n x[1-F(x)]dx - \frac{n[1-F(n)]}{d^2(\epsilon)},$$

hence

$$D_n + E_n \le \frac{2}{nd^2(\epsilon)}\int_0^n x[1-F(x)]dx + (1-\frac{1}{d^2(\epsilon)})n[1-F(n)].$$

If $\mu \le 1$ then $d(\epsilon)\le\mu\le 1$ for every $\epsilon>0$, and we get

$$D_n + E_n \le \frac{2}{nd^2(\epsilon)}\int_0^n x[1-F(x)]dx,$$

which gives the result since, as ϵ goes to $0, d(\epsilon)$ goes to μ and $\varphi(\epsilon)$ goes to 1.

If $\infty \ge EX_1 > 1$, there exists $\epsilon > 0$ such that $d(\epsilon) > 1$, hence if we observe that

$$n[1-F(n)] \le \frac{2}{n}\int_0^n x[1-F(x)]dx,$$

we get

$$D_n + E_n \le \frac{2}{n}\int_0^n x[1-F(x)]dx,$$

which gives the result. □

THEOREM 2. — *If* $\mu = EX_1 < \infty,$ *then*

$$\frac{2}{(1+\mu)^2} \le \liminf_{n\to\infty} \frac{R_n}{\frac{1}{n}\int_0^n x[1-F(x)]dx}.$$

PROOF. As in the proof of Theorem 3.1 it suffices to establish the inequality for the expression

$$A_n := n\int_0^\epsilon s\varphi''(s)\varphi^{n-1}(s)ds \ge n\int_0^\epsilon s\varphi''(s)\varphi^n(s)ds.$$

From lemma 3.4, for any $\delta > 0$ one can choose $\epsilon > 0$ such that $\varphi(s) \ge e^{-s(\mu+\delta)}$ for $s < \epsilon$. Hence

$$A_n \ge n\int_0^\epsilon s\varphi''(s)e^{-ns(\mu+\delta)}ds = n\int_0^\infty x^2 dF(x)\left(\int_0^\epsilon se^{-s[x+n(\mu+\delta)]}ds\right)$$
$$= n\int_0^\infty \left(\frac{x}{x+n(\mu+\delta)}\right)^2 dF(x)\int_0^{\epsilon[x+n(\mu+\delta)]} ye^{-y}dy \sim n\int_0^\infty \left(\frac{x}{x+n(\mu+\delta)}\right)^2 dF(x);$$

breaking the integral from 0 to n into the part D_n and from n to ∞ into the part E_n one obtains :

$$\text{a)}E_n := n\int_n^\infty \Big(\frac{x}{x+n(\mu+\delta)}\Big)^2 dF(x)$$
$$\geq n\int_n^\infty \Big(1-\frac{n(\mu+\delta)}{n+n(\mu+\delta)}\Big)^2 dF(x) \qquad = \frac{1}{[1+\mu+\delta]^2}n[1-F(n)];$$
$$\text{b)}\quad D_n := n\int_0^n \Big(\frac{x}{x+n(\mu+\delta)}\Big)^2 dF(x) \geq \frac{n}{[n+n(\mu+\delta)]^2}\int_0^n x^2 dF(x);$$

integration by parts yields :

$$D_n \geq \frac{2}{(1+\mu+\delta)^2}\frac{1}{n}\int_0^n x[1-F(x)]dx \;-\; \frac{1}{(1+\mu+\delta)^2}\; n\; [1-F(n)].$$

Finally, A_n is ultimately bounded below by $D_n + E_n$:

$$D_n + E_n \geq \frac{2}{(1+\mu+\delta)^2}\;\frac{1}{n}\;\int_0^n x[1-F(x)]dx,$$

from which the result follows since $\delta > 0$ can be chosen as small as we want. □

5. Asymptotic behaviour of R_n : special results.. —
References to Th. A. i refer to the appendix (section 6).

THEOREM 2. — *If X_1 belongs to the domain of attraction of a stable law of parameter $\alpha, 1 \leq \alpha < 2$ and $\mu = EX_1 < \infty$, then :*

$$lim_{n\to\infty}\frac{R_n}{n[1-F(n)]} = \frac{(2-\alpha)\;(1+\alpha)}{\mu^\alpha}.$$

PROOF. It suffices,from lemma 3.5, to establish the results for the expression :

$$B_n = \frac{1}{1-F(n)}\;\int_0^\epsilon s\varphi^{''}(s)\varphi^{n-1}(s)ds.$$

The change of variable $s = \frac{t}{n}$ yields

$$B_n = \frac{1}{1-F(n)}\frac{1}{n^2}\;\int_0^{n\epsilon} t\varphi^{''}(\frac{t}{n})\varphi^{n-1}(\frac{t}{n})dt.$$

Hence with the notations of Th. A.6 :

$$B_n = \alpha(2-\alpha)\underbrace{\frac{l(n)}{n^\alpha[1-F(n)]}}_{\to 1}\;\int_0^{n\epsilon} t^{\alpha-1}\underbrace{\frac{l(\frac{n}{t})}{l(n)}}_{\to 1}\underbrace{\frac{\varphi^{''}(\frac{t}{n})}{\alpha(2-\alpha)(\frac{n}{t})^{2-\alpha}l(\frac{n}{t})}}_{\to 1}\underbrace{\varphi^{n-1}(\frac{t}{n})}_{\to e^{-\mu t}}dt$$
$$= \int_0^\infty h_n(t)dt,$$

which defines h_n. From Th. A.6 we have, for every $t > 0$

$$\lim_{n\to\infty} h_n(t) = \alpha(2-\alpha)t^{\alpha-1}e^{-\mu t}.$$

Using Potter's theorem (Th A.1), Th. A.6 and lemma 3.3, the passage to the limit is justified by Lebesgue's dominated convergence theorem. Finally

$$\int_0^\infty h_n(t)dt \to \alpha(2-\alpha)\int_0^\infty t^{\alpha-1}e^{-\mu t}dt = \frac{(2-\alpha)(1+\alpha)}{\mu^\alpha}.$$

□

Theorem 2. — *If X_1 belongs to the domain of attraction of the normal law, then*

$$lim_{n\to\infty}\frac{R_n}{\frac{1}{n}\int_0^n x[1-F(x)]dx}=\frac{2}{\mu^2}.$$

PROOF. It suffices,from lemma 5, to establish the result for the expression :

$$C_n=\frac{1}{\frac{1}{n}\int_0^n x[1-F(x)]dx}\ n\int_0^{\epsilon} s\varphi^{''}(s)\varphi^{n-1}(s)ds.$$

The change of variable $s=\frac{t}{n}$ gives

$$C_n=\frac{1}{\int_0^n x[1-F(x)]dx}\int_0^{n\epsilon} t\varphi^{''}(\frac{t}{n})\varphi^{n-1}(\frac{t}{n})dt.$$

Using the notation of Th. A.5 :

$$C_n=\frac{l(n)}{\int_0^n x[1-F(x)]dx}\int_0^{n\epsilon} t\underbrace{\frac{l(\frac{n}{t})}{l(n)}}_{\to 1}\underbrace{\frac{\varphi^{''}(\frac{t}{n})}{l(\frac{n}{t})}}_{\to 1}\underbrace{\varphi^{n-1}(\frac{t}{n})}_{\to e^{-\mu t}}dt=\int_0^{\infty}h_n(t)dt,$$

which defines h_n. Integrating by parts and using part a) of Theorem A.5. gives :

$$\int_0^n x[1-F(x)]dx=\frac{[1-F(n)]n^2}{2}+\frac{l(n)}{2}.$$

Therefore

$$\frac{1}{l(n)}\int_0^n x[1-F(x)]dx=\frac{1}{2}\frac{n^2[1-F(n)]}{l(n)}+\frac{1}{2},$$

which tends to $\frac{1}{2}$ as $n\to\infty$, by Theorem A.5. As in the proof of the previous theorem, the result follows from Lebesgue's dominated convergence theorem : $\int_0^{\infty}h_n(t)dt\to 2\int_0^{\infty}te^{-\mu t}dt=\frac{2}{\mu^2}$. □

Theorems 5.1 and 5.2 are linked. For if the conditions of 5.1 are satisfied, then

$$\frac{n^2(1-F(n))}{\int_0^n x(1-F(x))dx}=\left\{\int_0^1 y\frac{1-F(ny)}{1-F(n)}dy\right\}^{-1}\ \to\ 2-\alpha.$$

Hence we obtain the following corollary :

Corollary 2. — *If X_1 belongs to the domain of attraction of a stable law of parameter $\alpha, 1\le\alpha\le 2$ and $\mu=EX_1<\infty$ then,*

$$lim_{n\to\infty}\frac{R_n}{\frac{1}{n}\int_0^n x[1-F(x)]dx}=\frac{(3-\alpha)\ (1+\alpha)}{\mu^{\alpha}}.$$

The following lemma will be used in the proofs of next theorems.

LEMMA 2. — *If X_1 belongs to the domain of attraction of a stable law of parameter $\alpha, 0 < \alpha \leq 1$, or if X_1 is relatively stable then*

i) There exists a sequence a_n such that $\varphi^n(\frac{s}{a_n}) \to e^{-s^\alpha}$ as $n \to \infty$, with $\alpha = 1$ if X_1 is relatively stable.

ii) There exists δ with $0 < \delta < 1$ and $\epsilon > 0$ such that :

$$\varphi^n\left(\frac{t}{a_n}\right) \leq e^{-At^\delta}, \quad 0 \leq t \leq a_n\epsilon, \quad a_n > \frac{1}{\epsilon}$$

PROOF. Consider first the case $0 < \alpha < 1$. The a_n are chosen as in the comments which follow Th A.6; by that theorem one has : $1 - F(x) \sim \frac{1}{(1-\alpha)}\frac{l(x)}{x^\alpha}$,with l slowly varying, which yields by Th. A.4. b) $1 - \varphi(s) = s^\alpha l(\frac{1}{s})$. Then $d(s) = \frac{1-\varphi(s)}{s} = s^{\alpha-1}l(\frac{1}{s})$ and lemma 3.3 yields :

$$\varphi(s) \leq e^{-sd(s)} \leq e^{-s^\alpha l(\frac{1}{s})}.$$

Therefore :

$$\varphi(\frac{s}{a_n}) \leq e^{-s^\alpha \frac{l(\frac{a_n}{s})}{a_n^\alpha}}, \quad \varphi^n(\frac{s}{a_n}) \leq e^{-s^\alpha \frac{l(\frac{a_n}{s})}{l(a_n)}\frac{nl(a_n)}{a_n^\alpha}}.$$

Now by (6.3), $\frac{nl(a_n)}{a_n^\alpha} \to 1$. The result follows by applying Th. A.1. (Potter's theorem) to the ratio $\frac{l(a_n)}{l(\frac{a_n}{s})}$, with $\epsilon = \frac{1}{X}$.

The case $\alpha = 1$ is treated in the same way by invoking Th. A.8 instead of Th. A.6. and A.4. Observe that $\widetilde{l}$ plays the role of l.

Finally when X_1 is relatively stable, the result follows from Th. A.9. $\square$

THEOREM 2. — *If X_1 belongs to the domain of attraction of a stable law of parameter $\alpha, 0 < \alpha < 1$, then*

$$lim_{n\to\infty}\ R_n \ = 1 - \alpha. \qquad (*)$$

Conversely if $(*)$ holds then X_1 belongs to the domain of attraction of a stable law of parameter $\alpha,\ 0 < \alpha < 1$.

PROOF. As before, to study the asymptotic behaviour of R_n it suffices,from lemma 3.5, to establish the results for A_n.

From the remark following Th. A.6, the assumption implies the existence of a sequence $a_n, a_n > 0$ such that $\varphi^n(\frac{s}{a_n}) \to e^{-s^\alpha}$ as $n \to \infty$. After the change of variable $s = \dfrac{t}{a_n}$, we get

$$A_n = \frac{n}{a_n^2}\int_0^{a_n\epsilon} t\varphi''(\frac{t}{a_n})\varphi^{n-1}(\frac{t}{a_n})dt.$$

Equations (6.4) and (6.5) of Theorem A.7 lead to :

$$\begin{aligned} A_n =& \frac{n}{a_n^2}\int_0^{a_n\epsilon} t\alpha(1-\alpha)(\frac{a_n}{t})^{2-\alpha}l(\frac{a_n}{t})\frac{\varphi''(\dfrac{t}{a_n})}{\alpha(1-\alpha)(\dfrac{a_n}{t})^{2-\alpha}l(\dfrac{a_n}{t})}\varphi^{n-1}(\frac{t}{a_n})dt \\ =& \underbrace{\frac{n}{a_n^\alpha}l(a_n)}_{\to 1}\alpha(1-\alpha)\int_0^{a_n\epsilon} t^{\alpha-1}\underbrace{\frac{l(\dfrac{a_n}{t})}{l(a_n)}}_{\to 1}\underbrace{\frac{\varphi''(\dfrac{t}{a_n})}{\alpha(1-\alpha)(\dfrac{a_n}{t})^{2-\alpha}\ l(\dfrac{a_n}{t})}}_{\to 1}\underbrace{\varphi^{n-1}(\frac{t}{a_n})}_{\to e^{-t^\alpha}}dt \\ =& \int_0^\infty h_n(t), \end{aligned}$$

which defines h_n. Formula (6.3) and Th. A.7 yields :

$$\lim_{n\to\infty} h_n(t) = \alpha(1-\alpha)t^{\alpha-1}e^{-t^\alpha}.$$

Let ϵ be chosen as in lemma 5.3 ; with the help of that lemma, the result follows from Lebesgue's dominated convergence theorem.

To prove the converse, let us assume that R_n converges to $c,\ 0 < c < 1$ as n goes to ∞. For any integer m we have

$$R_{nm+1} = (nm+1)\int_0^\infty s\varphi^{''}(s)\varphi^{nm}(s)ds = (nm+1)\int_0^\infty \varphi^{nm}(s)d(s),$$

where $(s) = \int_0^s t\varphi^{''}(t)dt$. The change of variable $\varphi(s) = 1-\frac{u}{n}, (s = \varphi^{-1}(1-\frac{u}{n}))$, yields

$$R_{nm+1} = (nm+1)\int_0^n (1-\frac{u}{n})^{nm} d\left(\varphi^{-1}(1-\frac{u}{n})\right).$$

Define

$$\mu_n(u) = n\left(\varphi^{-1}(1-\frac{u}{n})\right),$$

which are measures on $\mathcal{R}^+$ of mass n. Since $\mu_n(u) \le u$ weak compactness implies the existence of a subsequence $n_i \to \infty$ such that $\mu_{n_i}(u) \to \mu(u)$ weakly.

$$R_{nm+1} = \frac{(nm+1)}{n}\int_0^n (1-\frac{u}{n})^{nm} d\mu_n(u),$$

taking the limit along the subsequence n_i yields

$$c = m\int_0^\infty e^{-um}d\mu(u).$$

From

$$\int_0^\infty e^{-um}d\mu(u) = \frac{c}{m}, m = 1.2\ldots,$$

and the change of variable $x = e^{-u}$ it follows by Weierstrass'theorem that $\mu(u) = cu$, independently of the subsequence. Hence $\mu_n(u) \to cu$ weakly or

$$\mu_n(u) = n\left(\varphi^{-1}(1-\frac{u}{n})\right) \to cu \text{or } (s) \sim c[1-\varphi(s)] \text{as } s \to 0.$$

Thus

$$\frac{\int_0^s t\varphi^{''}(t)dt}{1-\varphi(s)} \to c \quad \text{or} \quad \int_0^s t\varphi^{''}(t)dt + c\int_0^s \varphi^{'}(t)dt = o(1-\varphi(s)).$$

Call the left hand side of this last equation $V(s)$:

$$V(s) := \int_0^s t\varphi''(t)dt + c\int_0^s \varphi'(t)dt = \int_0^s t^{1-c}[t^c\varphi'(t)]'dt$$

$$= s\varphi'(s) - (1-c)[\varphi(s)-1], \text{whence} \frac{V(s)}{1-\varphi(s)} = \frac{s\varphi'(s)}{1-\varphi(s)} + 1 - c = o(1), \text{as} s \to 0.$$

This yields $-\frac{s\varphi'(s)}{1-\varphi(s)} \to 1-c$ as $s \to 0$. Put $\delta(s) := 1-c+\frac{s\varphi'(s)}{1-\varphi(s),}$ to obtain

$$\int_t^1 \frac{\delta(s)}{s}ds = -(1-c)\log t + C + \log[1-\varphi(t)]$$

or $1-\varphi(t) = t^{-c+1}l(\frac{1}{t})$ where $l(u) = \exp\{-C+\int_1^u \frac{\epsilon(x)}{x}dx\}$ and $\epsilon(x) = \delta(\frac{1}{x})$. From the representation theorem (theorem A.2), it follows that l is a slowly varying function and then theorems A.5. and A.7. imply that F is attracted to a stable law of parameter $1-c$. $\square$

THEOREM 2. — *The following three statements are equivalent :*

(1) $$F \text{ is relatively stable,}$$

(2) $$\int_0^x [1 - F(y)]dy \quad \text{is a slowly varying function ,}$$

(3) $$lim_{n\to\infty} \ R_n \ = 0.$$

PROOF. (1) and (2) are equivalent by Th. A.9. Assume (2); as usual it suffices to establish the results for A_n.

Define $l(x) := \int_0^x (1 - F(y))\, dy$. After an integration by parts and a change of variables, one obtains :

$$\varphi''(s) = -\int_0^\infty e^{-sx} x^2 d(1-F) = \frac{1}{s}\int_0^\infty l(\frac{y}{s}) e^{-y}(2 - 4y + y^2)dy.$$

Replacing $\varphi''(s)$ by this value in the expression of A_n and using Fubini's theorem, one obtains :

$$A_n = \int_0^\infty e^{-y}(2 - 4y + y^2)\left[n\int_0^\epsilon \varphi^{n-1}(s) l(\frac{y}{s})ds\right]dy.$$

After the change of variables

$$s = \frac{t}{a_n} \quad \text{where} \quad \frac{nl(a_n)}{a_n} \sim 1 \quad \text{with} \quad \varphi^n(\frac{t}{a_n}) \to e^{-t},$$

$$\begin{aligned} A_n &= \int_0^\infty e^{-y}(2 - 4y + y^2)\left[\frac{n}{a_n}\int_0^{a_n\epsilon} \varphi^{n-1}(\frac{t}{a_n}) l(\frac{ya_n}{t})dt\right]dy \\ &= \int_0^\infty e^{-y}(2 - 4y + y^2)\left[\underbrace{\frac{nl(a_n)}{a_n}}_{\to 1}\int_0^{a_n\epsilon} \underbrace{\varphi^{n-1}(\frac{t}{a_n})}_{\to e^{-t}} \underbrace{\frac{l(\frac{ya_n}{t})}{l(a_n)}}_{\to 1} dt\right]dy. \end{aligned}$$

Let ϵ be chosen as in lemma 5.3; that lemma and the relation

$$\int_0^\infty e^{-y}(2 - 4y + y^2)dy = 0$$

then implies that $A_n \to 0$ as $n \to \infty$ by Lebesgue's dominated convergence theorem.

To prove the converse proceed as in theorem 5.3 with $c = 0$ to obtain $1 - \varphi(t) = tl(\frac{1}{t})$ with l a slowly varying function; it follows from Theorem A.4. ii) that $\int_0^x (1 - F(x))\, dx$ is a slowly varying function. $\square$

The last case to be considered is the one where X_1 belongs to the domain of attraction of a stable law of parameter α, with $\alpha = 1$ and $EX_1 = \infty$; then from Theorems A.6 and A.8 one has : $1 - F(x) \sim \frac{l(x)}{x}$, $x \to \infty$ and moreover one can choose a sequence $\{a_n\}_{n\geq 1}$, $a_n > 0$ such that $[\varphi(\frac{t}{a_n})]^n \to e^{-t}$, $n \to \infty$ and $\frac{\widetilde{l}(a_n)}{a_n} \sim \frac{1}{n}$ where $\widetilde{l}(x) = \int^x \frac{l(t)}{t}dt.$

THEOREM 2. — *If X_1 belongs to the domain of attraction of a stable law of parameter α, with $\alpha = 1$ and $EX_1 = \infty$ then*

$$lim_{n\to\infty}\frac{\widetilde{l}(a_n)}{l(a_n)}R_n = 1,$$

where the symbols $l, \widetilde{l}, a_n$ have been define above.

PROOF. As usual it suffices to establish the result for the expression : A_n.

After the change of variable $s = \dfrac{t}{a_n}$ we get

$$A_n = \frac{n}{a_n^2}\int_0^{a_n\epsilon} t\varphi''(\frac{t}{a_n})\varphi^{n-1}(\frac{t}{a_n})dt.$$

Using the notation of Th. A.8 :

$$A_n = \frac{l(a_n)}{\widetilde{l}(a_n)}\underbrace{\frac{n\widetilde{l}(a_n)}{a_n}}_{\to 1}\int_0^{n\epsilon}\underbrace{\frac{l(\frac{a_n}{t})}{l(a_n)}}_{\to 1}\underbrace{\frac{\varphi''(\frac{t}{a_n})}{\frac{a_n}{t}l(\frac{a_n}{t})}}_{\to 1}\underbrace{\varphi^{n-1}(\frac{t}{a_n})}_{\to e^{-t}}\,dt = \frac{l(a_n)}{\widetilde{l}(a_n)}\int_0^{\infty} h_n(t)dt,$$

which defines h_n and as in the previous theorems, with the help of lemma 5.3 the result follows from Lebesgue's dominated convergence theorem. □

REMARK 2. — *In the above theorem since $\frac{\widetilde{l}(a_n)}{l(a_n)} \to \infty$ one has the rate of convergence of R_n to* 0, *as $n \to \infty$.*

We thank J. Kuelbs for asking one of us a question which led to the following remark :

REMARK 2. — *If X_1 belongs to the domain of partial attraction of a stable law of parameter α, where $0 < \alpha < 1$, then, along a subsequence n_r, $\lim_{n_r\to\infty} R_n = 1-\alpha$. Hence if X_1 belongs to Doeblin's " universal law" the set of all limit points of R_n is the closed interval* $[0, 1]$.

Appendix. Slowly varying functions and domains of attraction. — Let l be a positive measurable function satisfying

$$\frac{l(\lambda x)}{l(x)} \to\ 1\ (x\to\infty) \quad \forall\lambda > 0 \tag{6.1}$$

then l is said to be slowly varying. If this is the case one can show that the convergence in (1) is uniform in each compact-λ set in $(0,\infty)$.

THEOREM 2. — *(i) If l is slowly varying then for any chosen constants $A > 1, \delta > 0$ there exists $X = X(A,\delta)$ such that*

$$\frac{l(y)}{l(x)} \le A\max\Big\{\ (\frac{y}{x})^{\delta}, (\frac{y}{x})^{-\delta}\ \Big\} \quad (x\ge X, y\ge X).$$

(ii) If, further, l is bounded away from 0 *and* ∞ *on every compact subset of* $[0,\infty)$, *then for every $\delta > 0$ there exists $A' = A'(\delta) > 1$ such that*

$$\frac{l(y)}{l(x)} \le A'\max\Big\{\ (\frac{y}{x})^{\delta}, (\frac{y}{x})^{-\delta}\ \Big\} \quad (x>0, y>0).$$

Theorem 2. — *The function l is slowly varying if and only if it may be written in the form*

$$l(x) = c(x)\ exp\{\int_1^x \frac{\epsilon(u)}{u} du\},$$

where $c(x) \to c \in (0,\infty)$, and $\epsilon(x) \to 0$ as $x \to \infty$.

In the sequel $F(x)$ will denote a distribution function whose support is $[0,\infty[$, $\hat{F}(s) = \varphi(s) = \int_0^\infty e^{-sx} dF(x), s \geq 0$, its Laplace transform, $V(x) = \int_0^x t^2 dF(t), x \geq 0$, the truncated variance of $F(\cdot)$ and

$$\hat{V}(s) = \varphi^{''}(s) = \int_0^\infty e^{-sx} dV(x) = \int_0^\infty e^{-sx} x^2 dF(x),\ s \geq 0.$$

Theorem 2. —
(i) If l is slowly varying, $0 \leq \alpha < 2$ then the following two statements are equivalent :
a) $V(x) \sim\ x^{2-\alpha} l(x)\ \ ,\ x \to \infty$,
b) $1 - F(x) \sim\ \frac{2-\alpha}{\alpha} \frac{l(x)}{x^\alpha}\ ,\ x \to \infty$.
(ii) If l is slowly varying, $(\alpha = 2)$, then the following two statements are equivalent :
a) $V(x) \sim l(x), x \to \infty$,
b) $\frac{x^2[1-F(x)]}{V(x)} \to 0, x \to \infty$.

Theorem 2. — *(i) If l is slowly varying $\ 0 \leq \alpha < 1$ then the following two statements are equivalent :*
a) $1 - F(x) \sim\ \frac{1}{(1-\alpha)} \frac{l(x)}{x^\alpha}$, $x \to \infty$
b) $1 - \hat{F}(s) \sim\ s^\alpha l(\frac{1}{s})$, $s \to 0$.
(ii) The following two statements are equivalent $(\alpha = 1)$:
a) $\int_0^x [1 - F(x)] \sim l(x), x \to \infty$
b) $1 - \hat{F}(s) \sim\ sl(\frac{1}{s})$, $s \to 0$.

The following two theorems on domains of attractions are an adaptation of Th. 8.3.1. of [BGT] when the support of F is $\mathcal{R}^+$:

Theorem 2. — *the following four statements are equivalent :*
a) F is attracted to a normal law $(\alpha = 2)$,
b) $V(x) = l(x)$ where $l(x)$ is slowly varying,
c) $\frac{x^2[1-F(x)]}{V(x)} \to 0$ as $x \to \infty$,
d) $\hat{V}(s) = \varphi^{''}(s) \sim l(\frac{1}{s})$ as $s \to 0$.

Theorem 2. — *the following four statements are equivalent :*
a) F is attracted to a stable law $(0 < \alpha < 2)$,
b) $1 - F(x) \sim \frac{l(x)}{x^\alpha}$ where $l(x)$ is slowly varying ,
c) $V(x) \sim \frac{\alpha}{2-\alpha} x^{2-\alpha} l(x)$ as $x \to \infty$,
d) $\hat{V}(s) = \varphi^{''}(s) \sim \alpha(2-\alpha) \frac{1}{s^{2-\alpha}} l(\frac{1}{s})$ as $s \to 0$.

We recall from [BGT] section (8.3.5.) that the stable laws with Laplace transform e^{-s^α}, $o < \alpha < 1$, are (to within scale) the only ones concentrated on $[0,\infty[$. Consider their domain of attraction and observe that no centering is required : F is in their domain of attraction if and only if there exists a sequence a_n, with $a_n > 0$ and $\frac{a_n}{a_{n+1}} \to 1$ such that $\varphi^n(\frac{s}{a_n}) \to e^{-s^\alpha}$ as $n \to \infty$. Moreover one has

$$1 - F(x) \sim \frac{1}{(1-\alpha)} \frac{l(x)}{x^\alpha}, \quad \text{for some slowly varying function} l, \tag{6.2}$$

and the norming constants are determined from

$$\frac{nl(a_n)}{a_n^\alpha} \to 1. \tag{6.3}$$

THEOREM 2. — *Equation (6.2) is equivalent to each of the following :*

$$V(x) \sim \frac{\alpha}{(2-\alpha)(1-\alpha)} x^{2-\alpha} l(x), \quad x \to \infty. \tag{6.4}$$

$$\hat{V}(s) = \varphi''(s) \sim \alpha(1-\alpha)(\frac{1}{s})^{2-\alpha} l(\frac{1}{s}), \quad s \to 0. \tag{6.5}$$

When $\alpha = 1$ the situation is slightly more technical : recall from [BGT] that the class $_l$ of $l-$index equals to 1 is the class of functions f such that for $\lambda > 0$,

$$lim_{x\to\infty} \frac{\{f(\lambda x) - f(x)\}}{l(x)} = \log \lambda.$$

If $l(x)$ is slowly varying, then

$$\widetilde{l}(x) = \int^x \frac{l(t)}{t} dt$$

is slowly varying and

$$\frac{\widetilde{l}(x)}{l(x)} \to \infty \quad \text{as} x \to \infty.$$

THEOREM 2. — *If l is slowly varying then the following four statements are equivalent :*

$$a)\ 1 - F(x) \sim \frac{l(x)}{x}, \quad x \to \infty$$

$$b)\ \varphi''(s) \sim \frac{1}{s} l(\frac{1}{s}), s \to 0$$

$$c)\ \frac{1-\varphi(s)}{s} \in {}_l \ of\ l - index\ equals\ \ to\ 1$$

$$d)\ \frac{1-\varphi(s)}{s} \sim \int_1^{\frac{1}{s}} \frac{l[t]}{t} dt = \widetilde{l}(\frac{1}{s}), \quad s \to 0.$$

Moreover if F is in the domain of attraction of a stable law of parameter 1*, one can choose a sequence $\{a_n\}_{n\geq 1}$, $a_n > 0$ such that*

$$\frac{\widetilde{l}(a_n)}{a_n} \sim \frac{1}{n} \quad \textit{so that } [\varphi(\frac{t}{a_n})]^n \to e^{-t} \quad , \quad n \to \infty$$

PROOF the equivalence of a) and b) is Th. A.6 for $\alpha = 1$. This is equivalent to c) by the remark in [BGT] p.335 and the last two statements are equivalent by Th. 3.6.6 in [BGT] p.158. Finally the last statement follows from Th. 8.8.1 of [BGT] pp.373-74 since one can choose a sequence a_n such that $\frac{\widetilde{l}(a_n)}{a_n} \sim \frac{1}{n}$, this yield the conclusion. □

Recall that a law F is relatively stable if there exists norming constants a_n such that $\lim \varphi^n(\frac{s}{a_n}) = e^{-s}$. The following theorem is an adaptation of Theorem 8.8.1. and formula (8.8.1) of [BGT].

THEOREM 2. — *The following three statements are equivalent :*
i) F *is relatively stable,*
ii) $\int_0^x (1 - F(y))dy \sim l(x), x \to \infty$ *where* l *is slowly varying,*
iii) $1 - \varphi(s) \sim sl(\frac{1}{s}), s \to 0$ *where* l *is slowly varying.*
Moreover if F *is relatively stable the norming constants* a_n *satisfy :*

$$\frac{l(a_n)}{a_n} \sim \frac{1}{n}$$

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