## 29<sup>th</sup> International Mathematical Olympiad Canberra, Australia Day I

- 1. Consider two coplanar circles of radii R and r (R > r) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then A = P.)
  - (i) Find the set of values of  $BC^2 + CA^2 + AB^2$ .
  - (ii) Find the locus of the midpoint of BC.
- 2. Let n be a positive integer and let  $A_1, A_2, \ldots, A_{2n+1}$  be subsets of a set B. Suppose that
  - (a) Each  $A_i$  has exactly 2n elements,
  - (b) Each  $A_i \cap A_j$   $(1 \le i < j \le 2n+1)$  contains exactly one element, and
  - (c) Every element of B belongs to at least two of the  $A_i$ .

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that  $A_i$  has 0 assigned to exactly n of its elements?

3. A function f is defined on the positive integers by

$$f(1) = 1, f(3) = 3,$$
  

$$f(2n) = f(n),$$
  

$$f(4n+1) = 2f(2n+1) - f(n),$$
  

$$f(4n+3) = 3f(2n+1) - 2f(n),$$

for all positive integers n.

Determine the number of positive integers n, less than or equal to 1988, for which f(n) = n.

## 29<sup>th</sup> International Mathematical Olympiad Canberra, Australia Day II

4. Show that set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \ge \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

- 5. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that  $S \ge 2T$ .
- 6. Let a and b be positive integers such that ab + 1 divides  $a^2 + b^2$ . Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.